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On the Orlicz–Sobolev Imbedding Theorem

R. A. ADAMS*

Department of Mathematics, University of British Columbia, Vancouver, Canada V6T 1W5

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The Orlicz space analog of the Sobolev imbedding theorem established for bounded domains by Donaldson and Trudinger is here extended to unbounded domains.

1. INTRODUCTION

If, in the definition of the ordinary Sobolev space $W^{m,p}(\Omega)$, the role played by the Lebesgue space $L^p(\Omega)$ is assumed instead by a more general Orlicz space $L_A(\Omega)$, the resulting structure is called an Orlicz–Sobolev space and denoted $W^m L_A(\Omega)$. In their paper [5], Donaldson and Trudinger have shown that Orlicz–Sobolev spaces possess many of the most useful properties of ordinary Sobolev spaces. In particular, they established imbeddings and compact imbeddings of Orlicz–Sobolev spaces analagous to those imbedding and compact imbedding results for ordinary Sobolev spaces commonly referred to under the blanket titles, "the Sobolev imbedding theorem" and "the Rellich–Kondrachov compactness theorem." For all of their results Donaldson and Trudinger assume the boundedness of the domain Ω over which their function spaces are defined. This boundedness is a natural requirement for the compactness theorem but it is not natural and, as we shall show below, not necessary for existence of the imbeddings.

In the usual proof of the Sobolev imbedding theorem (see [2, 6]) the key imbeddings are obtained for unbounded domains by "piecing together" results already obtained for bounded domains. This technique is not appropriate where Orlicz–Sobolev spaces are concerned, because the defining N-function $A(t)$ for an Orlicz space $L_A(\Omega)$ may behave very differently for small values of t than it does for large values. (By contrast, the defining function $A(t) = t^p$ for the Lebesgue space $L^p(\Omega)$ is homogeneous for all positive t.) In their proof of the Orlicz–Sobolev imbedding theorem Donaldson and Trudinger obtain certain

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growth estimates for the N-functions involved which require the boundedness of Ω for effective application. We shall prove the key imbeddings by the same technique, but we sharpen the growth estimate sufficiently so that boundedness is no longer required.

Allowing unbounded domains, however, introduces other complications into the imbedding picture. Because they are concerned only with bounded domains, Donaldson and Trudinger are able to assume certain behavior of their N-functions near zero—they are free to redefine the N-functions near zero if necessary, for such redefinition will not change the corresponding Orlicz spaces if the domain has finite volume. We shall have to work somewhat harder to obtain the appropriate imbeddings if the N-functions do not have the requisite behavior near zero. (For certain "trace" imbeddings, the imbedding problem in this case is still open.)

Our imbedding results are stated in Sections 3 (first-order imbeddings) and 5 (higher-order imbeddings). In Section 6 we quote an appropriate compact imbedding theorem. Section 4 is devoted to proving the assertions of Section 3. Section 7 points out some gaps and unanswered questions concerning the imbedding of Orlicz–Sobolev spaces. Section 2 below is devoted to recounting some of the basic definitions and properties of Sobolev, Orlicz, and Orlicz–Sobolev spaces.

2. Sobolev, Orlicz, and Orlicz–Sobolev Spaces

Throughout this paper Ω denotes a domain (open set) in Euclidean n-space, R^n. For positive integral m and real $p \geqslant 1$, $W^{m,p}(\Omega)$ denotes the "usual" Sobolev space consisting of (equivalence classes of) functions $u \in L^p(\Omega)$ with distributional derivatives $D^\alpha u$ of orders $|\alpha| \leqslant m$ also belonging to $L^p(\Omega)$; $W^{m,p}(\Omega)$ is a Banach space with norm

$$\| u \|_{m,p,\Omega} = \left\{ \sum_{|\alpha| \leqslant m} \| D^\alpha u \|_{p,\Omega}^p \right\}^{1/p},$$

$\| \cdot \|_{p,\Omega}$ being the norm in $L^p(\Omega)$. The basic properties of these spaces can be found, for example, in [2].

If, in the definition of the Lebesgue space $L^p(\Omega)$, the role played by the convex function t^p is assumed by a more general convex "Young's function" $A(t)$, (see, for example, [9]), the resulting space is called an Orlicz space. We shall assume that $A(t)$ belongs to the slightly more restrictive class of "N-functions" for the definition and basic properties of which the reader is referred to Krasnosel'skii and Rutickii [8], Donaldson and Trudinger [5], or Adams [2]. We shall denote the complement of a given N-function A by $\tilde{A}$,

$$\tilde{A}(s) = \max_{t \geq 0} (st - A(t)) \qquad (s \geqslant 0),$$

and shall make use of the inequalities

$$st \leqslant A(t) + \tilde{A}(s) \qquad \text{(Young's inequality)},$$

and

$$t < A^{-1}(t)\,\tilde{A}^{-1}(t) \leqslant 2t \qquad (t > 0).$$

If there exists a positive constant k such that

$$A(2t) \leqslant kA(t) \tag{1}$$

holds for all $t \geqslant 0$ (resp., for all $t \geqslant t_0 > 0$), then A is said to satisfy a global Δ_2 condition (resp., a Δ_2 condition near infinity). The pair (A, Ω) is called Δ-regular if either A satisfies a global Δ_2 condition or else A satisfies a Δ_2 condition near infinity and Ω has finite volume.

Conforming with standard usage, we denote by $K_A(\Omega)$ the "Orlicz class" of functions u defined a.e. on Ω such that $\int_\Omega A(|u(x)|)\,dx < \infty$. Always a convex set, $K_A(\Omega)$ is a vector space if and only if (A, Ω) is Δ-regular. The Orlicz space $L_A(\Omega)$ is the linear hull of $K_A(\Omega)$ and is a Banach space with the "Luxemburg norm"

$$\| u \|_{A,\Omega} = \inf \left\{ \lambda > 0 \colon \int_\Omega A(u(x)/\lambda)\,dx \leqslant 1 \right\}.$$

The generalized Hölder inequality,

$$\left| \int_\Omega u(x)\,v(x)\,dx \right| \leqslant 2 \| u \|_{A,\Omega} \| v \|_{\tilde{A},\Omega},$$

is a consequence of Young's inequality.

The Orlicz space $E_A(\Omega)$ is the closure in $L_A(\Omega)$ of the class of functions u which are bounded on Ω and have bounded supports in $\bar{\Omega}$. $E_A(\Omega)$ is the maximal linear subspace of $K_A(\Omega)$,

$$E_A(\Omega) \subset K_A(\Omega) \subset L_A(\Omega),$$

both containments being strict if (A, Ω) is not Δ-regular and equalities otherwise.

The Orlicz–Sobolev space $W^m L_A(\Omega)$ [resp., $W^m E_A(\Omega)$] consists of those (equivalence classes of) functions $u \in L_A(\Omega)$ [resp., $u \in E_A(\Omega)$] for which $D^\alpha u \in L_A(\Omega)$ [resp., $E_A(\Omega)$] for $|\alpha| \leqslant m$. $W^m L_A(\Omega)$ is a Banach space with respect to the norm

$$\| u \|_{m,A,\Omega} = \max_{|\alpha| \leqslant m} \| D^\alpha u \|_{A,\Omega},$$

and $W^m E_A(\Omega)$ is a closed subspace of $W^m L_A(\Omega)$, proper if and only if (A, Ω) is not Δ-regular.

$W_0^m L_A(\Omega)$ is the closure in $W^m L_A(\Omega)$ of the space $C_0^\infty(\Omega)$ of infinitely differentiable functions having compact supports in Ω.

3. The Sobolev and Orlicz–Sobolev Imbedding Theorems

Most of the imbeddings supplied by these theorems obtain for domains having the cone property. Ω has the cone property if each of its points is the vertex of a finite right-spherical cone contained in Ω and congruent to some fixed finite right-spherical cone. Certain imbeddings which we include as part of the theorems, but which will not directly concern us in this paper, require that Ω have the somewhat stronger "strong local Lipschitz property" (see [2, Sect. 4.5]).

The imbedding theorems assert the existence of imbeddings (continuous injections) of Sobolev and Orlicz–Sobolev spaces into Lebesgue and Orlicz spaces as well as spaces of the following types:

(i) spaces $C_B{}^j(\Omega)$ consisting of functions u which are j times continuously differentiable on Ω and for which $D^\alpha u(x)$ is bounded on Ω for $|\alpha| \leqslant j$. $C_B{}^j(\Omega)$ is a Banach space with norm

$$\| u \|_{C_B{}^j(\Omega)} = \max_{0 \leqslant |\alpha| \leqslant j} | D^\alpha u(x)|.$$

(ii) Spaces $L^q(\Omega_r)$ and $L_A(\Omega_r)$ where Ω_r denotes the intersection of Ω with an r-dimesnsional plane in R^n, considered as a domain in R^k.

We denote an imbedding of X into Y by the notation $X \to Y$; this implies that

(a) $X \subset Y$, and

(b) $\| u \|_Y \leqslant K \| u \|_X$ for all $u \in X$, where the constant K is independent of u. (In the case of imbeddings of Sobolev or Orlicz–Sobolev spaces into spaces of types (i) or (ii) above, condition (a) is weakened to assert that some element in the equivalence class constituting u belongs to Y.)

Theorem 3.1 (the Sobolev imbedding theorem). *Let Ω be a domain in R^n having the cone property and let Ω_r be the intersection of Ω with an r-dimensional plane in R^n.*

(a) *If $mp < n$, then $W^{m,p}(\Omega) \to L^q(\Omega)$ for $p \leqslant q \leqslant np/(n - mp)$.*

(b) *If $mp < n$ and $n - mp < r \leqslant n$, then $W^{m,p}(\Omega) \to L^q(\Omega_r)$ for $p \leqslant q \leqslant rp/(n - mp)$ (if $p = 1$,* (b) *also holds for $r = n - m$).*

(c) *If $mp = n$ and $1 \leqslant r \leqslant n$, then $W^{m,p}(\Omega) \to L^q(\Omega_r)$ for $p \leqslant q < \infty$ (if $p = 1$, $m = n$,* (c) *holds for $q = \infty$ as well; in fact, $W^{n,1}(\Omega) \to C_B{}^0(\Omega)$).*

(d) *If $mp > n$, then $W^{m,p}(\Omega) \to C_B{}^0(\Omega)$. In fact, if $(m - j)\, p > n$, then $W^{m,p}(\Omega) \to C_B{}^j(\Omega)$. If Ω has the strong local Lipschitz property then* (d) *can be strengthened to yield Hölder continuity estimates: If $(m - j)\, p > n > (m - j - 1)$ then, for all $x, y \in \Omega$,*

$$\frac{| D^\alpha u(x) - D^\alpha u(y)|}{| x - y |^\lambda} \leqslant K \| u \|_{m,p,\Omega} ,$$

whenever $|\alpha| \leqslant j$ *and* $0 < \lambda \leqslant m - j - (n/p)$ (*if* $n = (m - j - 1)p$, *the same result holds with* $0 < \lambda < 1$, *or* $0 < \lambda \leqslant 1$ *if* $p = 1$).

A complete proof of Theorem 3.1 can be found in [2]. Note that in case (c) there is no optimal (smallest) target space for the imbedding. In [10] Trudinger has shown that for bounded Ω and $mp = n$,

$$W^{m,p}(\Omega) \to L_A(\Omega),$$

where $A(t) = \exp(t^{p/(p-1)}) - 1$, and in [2] this result is extended to unbounded Ω provided $A(t)$ is replaced by

$$A_0(t) = \exp(t^{p/(p-1)}) - \sum_{j=0}^{k-1} (1/j!)\, t^{jp/(p-1)},$$

where k is the smallest integer $\geqslant p - 1$.

In generalizing the Sobolev imbedding theorem to cover imbeddings of $W^m L_A(\Omega)$ it is simplest to consider first the case $m = 1$.

If A is an N-function which satisfies

$$\int_0^1 \frac{A^{-1}(\tau)}{\tau^{(n+1)/n}}\, d\tau < \infty, \tag{2}$$

$$\int_1^\infty \frac{A^{-1}(\tau)}{\tau^{(n+1)/n}}\, d\tau = \infty, \tag{3}$$

the Sobolev conjugate N-function A_* is defined by

$$A_*^{-1}(t) = \int_0^t \frac{A^{-1}(\tau)}{\tau^{(n+1)/n}}\, d\tau. \tag{4}$$

In [5], Donaldson and Trudinger prove the following theorem.

THEOREM 3.2 (first-order Orlicz–Sobolev imbedding theorem for bounded domains). *Let* Ω *be a bounded domain in* R^n *having the cone property. Let* Ω_r *denote the intersection of* Ω *with an r-dimensional plane in* R^n.

(a) *If* A *satisfies* (2) *and* (3), *then* $W^1L_A(\Omega) \to L_{A_*}(\Omega)$.

(b) *If* A *satisfies* (2) *and* (3), *and if there exists* p *satisfying* $1 \leqslant p < n$ *such that* $B(t) = A(t^{1/p})$ *is an* N-*function, and if* $n - p < r \leqslant n$, *or* $p = 1$ *and* $n - 1 \leqslant r \leqslant n$, *then* $W^1L_A(\Omega) \to L_{A_*^{r/n}}(\Omega_r)$ *where* $A_*^{r/n}(t) = [A_*(t)]^{r/n}$.

(c) *If* $\int_1^\infty A^{-1}(\tau)\,\tau^{-(n+1)/n}\, d\tau < \infty$, *then* $W^1L_A(\Omega) \to C_B{}^0(\Omega)$. *If, also,* Ω *has the strong local Lipschitz property, then* (c) *can be stengthened to yield, for all* x, $y \in \Omega$,

$$|u(x) - u(y)| \leqslant K \|u\|_{1,A,\Omega} \int_{|x-y|^{-n}}^\infty \frac{A^{-1}(\tau)}{\tau^{(n+1)/n}}\, d\tau.$$

Note that (c) *implies* $W^1L_A(\Omega) \to L_B(\Omega)$ *for any* N-*function* B.

Although both (2) and (3) are assumed in parts (a) and (b), only (3) is a serious hypothesis. Since Ω is bounded and so has finite volume, we can always redefine any given N-function A near zero so that (2) holds, and, in so doing, will not change the space $W^1L_A(\Omega)$ except to replace its norm with an equivalent norm. This situation no longer obtains for unbounded Ω so in our generalization of part (a) below we will not assume that (2) is satisfied. However, it must be born in mind that a new definition of A_* will have to be given in this case—the definition (4) makes sense only if (2) is satisfied. (The new definition is given in Section 4 below (Definition 4.6).) Unfortunately, we cannot dispense with (2) as a hypothesis for (b).

THEOREM 3.3 (first-order Orlicz–Sobolev imbedding theorem). *Let Ω be a domain in R^n having the cone property. Let Ω_r denote the intersection of Ω with an r-dimensional plane in R^n.*

(a) *If* $\int_1^\infty A^{-1}(\tau)\,\tau^{-(n+1)/n}\,d\tau = \infty$, *then* $W^1L_A(\Omega) \to L_{A_*}(\Omega)$ *and* $W^1E_A(\Omega) \to E_{A_*}(\Omega)$.

(b) *If* $\int_1^\infty A^{-1}(\tau)\,\tau^{-(n+1)/n}\,d\tau = \infty$ *and* (2) *is satisfied, and if there exists p such that $1 \leqslant p < n$ and $B(t) = A(t^{1/p})$ is an N-function, and if $n - p < r \leqslant n$, or $p = 1$ and $n - 1 \leqslant r \leqslant n$, then* $W^1L_A(\Omega) \to L_{A_*^{r/n}}(\Omega)$ *and* $W^1E_A(\Omega) \to E_{A_*^{r/n}}(\Omega)$.

(c) (*Identical to Theorem* 3.2(c)).

If Ω is unbounded, (c) *implies that* $W^1L_A(\Omega) \to L_B(\Omega)$ *and* $W^1E_A(\Omega) \to E_B(\Omega)$ *for any N-function B satisfying $B(t) \leqslant KA(t)$ for all sufficiently small t.*

The proof of part (c) as given by Donaldson and Trudinger extends without complication to unbounded domains Ω, so we shall omit (c) from further consideration here. The proofs of (a) and (b) are given in the next section. Theorem 3.3 is generalized to higher-order spaces ($W^mL_A(\Omega)$) in Section 5.

4. PROOF OF THE IMBEDDING THEOREM

The proof will be carried out in several lemmas. The first of these establishes an estimate for the Sobolev conjugate N-function A_* defined by (4).

LEMMA 4.1. *Let A be an N-function satisfying* (2) *and* (3) *and suppose that, for some p such that $1 \leqslant p < n$, the function B defined by $B(t) = A(t^{1/p})$ is an N-function. Let $q = np/(n - p)$ and let A_* be defined by* (4). *Then the following conclusions may be drawn.*

(a) *For any $\lambda \geqslant 1$ the function $A_*^{\lambda p/q}$ is an N-function; in particular, A_* is an N-function;*

(b) $[A_*(t)]^{p/q} \leqslant q^{-p} A(t)$ *for all* $t \leqslant A_*^{-1}(1)$;

(c) *for every* $\epsilon > 0$ *there exists a constant* K_ϵ *such that for every* t

$$[A_*(t)]^{p/q} \leqslant \epsilon A_*(t) + K_\epsilon A(t). \tag{5}$$

Remark. In place of (5), Donaldson and Trudinger used a somewhat weaker estimate

$$[A_*(t)]^{p/q} \leqslant \epsilon A_*(t) + K_\epsilon t.$$

In the context of their proof this weaker estimate could only be used if Ω had finite volume.

Proof. Let $Q(t) = [A_*(t)]^{\lambda p/q}$. Noting that $B^{-1}(t) = [A^{-1}(t)]^p$, we readily calculate that

$$\begin{aligned}(Q^{-1})'(t) &= \frac{d}{dt} A_*^{-1}(t^{q/\lambda p}) = \frac{q}{\lambda p} \frac{A^{-1}(t^{q/\lambda p})}{t^{1+(q/n\lambda p)}} \\ &= \frac{q}{\lambda p} \left[\frac{B^{-1}(t^{q/\lambda p})}{t^{q/\lambda p}}\right]^{1/p} t^{-\mu},\end{aligned}$$

where $\mu = 1 + (q/n\lambda p) - (q/\lambda p^2) \geqslant 0$, since $\lambda \geqslant 1$ and $1 \leqslant p < n$. Being the inverse of an N-function, B^{-1} satisfies $\lim_{\tau\to 0+} B^{-1}(\tau)/\tau = \infty$ and $\lim_{\tau\to\infty} B^{-1}(\tau)/\tau = 0$. Moreover, B^{-1} is concave so that if $0 < \tau < \sigma$, then $B^{-1}(\tau)/B^{-1}(\sigma) \geqslant \tau/\sigma$. Hence, if $0 < t < s$,

$$\frac{(Q^{-1})'(t)}{(Q^{-1})'(s)} \geqslant \left(\frac{s}{t}\right)^{\mu} \geqslant 1.$$

It follows that $(Q^{-1})'(t)$ is positive and decreases monotonically from ∞ to 0 as t increases from 0 to ∞, so that Q is an N-function and (a) is proved.

We use a similar concavity argument to establish (b). If $t \leqslant 1$, then certainly $t \leqslant q^p t^{p/q}$ and we have

$$\begin{aligned}A_*^{-1}(t) &= \int_0^t \frac{[B^{-1}(\tau)]^{1/p}}{\tau^{(n+1)/n}}\, d\tau \\ &\geqslant \left[\frac{B^{-1}(q^p t^{p/q})}{q^p t^{p/q}}\right]^{1/p} \int_0^t \tau^{(1/p)-(1/n)-1}\, d\tau \\ &= A^{-1}(q^p t^{p/q}).\end{aligned}$$

Hence $t^{p/q} \leqslant q^{-p} A(A_*^{-1}(t))$, and (b) follows if we replace t by $A_*(t)$.

Now let $g(t) = A_*(t)/A(t)$ and $h(t) = [A_*(t)]^{p/q}/A(t)$. Clearly $g(t)/h(t) = [A_*(t)]^{p/n}$ tends to infinity with t. Hence there exists t_0 such that if $t \geqslant t_0$ then $h(t) \leqslant \epsilon g(t)$; i.e.,

$$[A_*(t)]^{p/q} \leqslant \epsilon A_*(t) \qquad \text{if} \quad t \geqslant t_0\,.$$

By (b), h is bounded on $[0, t_0]$ by some constant K_ϵ and (c) follows at once. ∎

LEMMA 4.2. *If $u \in W^{1,1}_{\text{loc}}(\Omega)$ (i.e., if the restriction of u to G belongs to $W^{1,1}(G)$ for every subdomain G whose closure is a compact subset of Ω), and if f satisfies a Lipschitz condition on R then the composition $f \circ u$ belongs to $W^{1,1}_{\text{loc}}(\Omega)$ and the chain rule holds*:

$$D_j(f \circ u)(x) = f'(u(x))\, D_j u(x).$$

In particular,

(i) *if $u \in W^{1,1}(\Omega)$, then $f \circ u \in W^{1,1}(\Omega)$;*

(ii) *if $u \in W^1 L_A(\Omega)$, then $f \circ u \in W^1 L_A(\Omega)$;*

(iii) *if $u \in W^{1,1}_{\text{loc}}(\Omega)$, then $f \circ | u | \in W^{1,1}_{\text{loc}}(\Omega)$.*

This result is standard. The proof is essentially identical to that of [2, Lemma 8.31], and we omit it here.

We are now ready to prove part (a) of Theorem 3.3 provided (2) is satisfied. The technique is that of Donaldson and Trudinger.

LEMMA 4.3. *Let Ω be a domain in R^n having the cone property, and let A be an N-function satisfying* (2) *and* (3) *so that its Sobolev conjugate A_* is defined by* (4). *Then*

$$W^1 E_A(\Omega) \to E_{A_*}(\Omega),$$
$$W^1 L_A(\Omega) \to L_{A_*}(\Omega).$$

Proof. Let $u \in W^1 L_A(\Omega)$ and suppose, for the moment, that u is bounded and has bounded support, and that u is not 0 in $L_A(\Omega)$. Then $\int_\Omega A_*(| u(x)|/\lambda)\, dx$ decreases continuously from infinity to zero as λ increases from zero to infinity, so that

$$\int_\Omega A_*\left(\frac{| u(x)|}{U}\right) dx = 1 \qquad \text{where} \quad U = \| u \|_{A_*,\Omega}\,.$$

We are looking for a constant K, independent of u, such that $U \leqslant K \| u \|_{1,A,\Omega}$; accordingly we may assume, without loss of generality, that $\| u \|_{A,\Omega} \leqslant U$.

Now let $\omega(t) = [A_*(t)]^{1/q}$, where $q = n/(n-1)$, and set $f(x) = \omega(| u(x)|/U)$. Evidently $u \in W^{1,1}(\Omega)$ and ω satisfies a Lipschitz condition on the range of $| u |/U$ (by Lemma 4.1, case $p = 1$), and so, by Lemma 4.2, $f \in W^{1,1}(\Omega)$. Since Ω has the cone property, there is a constant K_1 (independent of u) such that

$$\begin{aligned} 1 &= \left\{\int_\Omega A_*\left(\frac{| u(x)|}{U}\right) dx\right\}^{1/q} = \| f \|_{q,\Omega} \leqslant K_1 \| f \|_{1,1,\Omega} \\ &= K_1 \left\{\sum_{j=1}^n \frac{1}{U} \int_\Omega \left| \omega'\left(\frac{| u(x)|}{U}\right)\right| | D_j u(x)|\, dx + \int_\Omega \omega\left(\frac{| u(x)|}{U}\right) dx\right\} \\ &\leqslant \frac{2K_1}{U} \sum_{j=1}^n \left\| \omega'\left(\frac{| u |}{U}\right)\right\|_{\tilde{A},\Omega} \| D_j u \|_{A,\Omega} + K_1 \int_\Omega \omega\left(\frac{| u(x)|}{U}\right) dx, \end{aligned} \tag{6}$$

where the generalized Hölder inequality has been used in the last line.

Now the function $s = A_*(t)$ satisfies the differential equation (obtained from (4))

$$A^{-1}(s)(ds/dt) = s^{(n+1)/n}, \tag{7}$$

and hence, since $\tilde{A}^{-1}(s)\, A^{-1}(s) \geqslant s$; also the inequality

$$ds/dt \leqslant s^{1/n} \tilde{A}^{-1}(s).$$

Hence $\omega(t)$ satisfies the inequality

$$\omega'(t) \leqslant (1/q)\, \tilde{A}^{-1}(A_*(t)).$$

If $\lambda \geqslant 1$, we have

$$\int_\Omega A\left(\frac{\tilde{A}^{-1}(A_*(|\,u(x)|/U))}{\lambda}\right) dx \leqslant \frac{1}{\lambda} \int_\Omega A_*\left(\frac{|\,u(x)|}{U}\right) dx \leqslant 1.$$

It follows that

$$\left\| \omega'\left(\frac{|\,u\,|}{U}\right)\right\|_{\tilde{A},\Omega} \leqslant \frac{1}{q} \left\| \tilde{A}^{-1}\left(A_*\left(\frac{|\,u\,|}{U}\right)\right)\right\|_{\tilde{A},\Omega} \leqslant \frac{1}{q}\,. \tag{8}$$

Moreover, by Lemma 4.1(c), with $p = 1$ and $\epsilon = 1/(2K_1)$, we have, for some constant K_2 independent of u,

$$\begin{aligned} \int_\Omega \omega\left(\frac{|\,u(x)|}{U}\right) dx &\leqslant \frac{1}{2K_1} \int_\Omega A_*\left(\frac{|\,u(x)|}{U}\right) dx + K_2 \int_\Omega A\left(\frac{|\,u(x)|}{U}\right) dx \\ &\leqslant \frac{1}{2K_1} + \frac{K_2}{U} \|\,u\,\|_{A,\Omega}\,, \end{aligned} \tag{9}$$

since we have assumed $\|\,u\,\|_{A,\Omega} \leqslant U$. Combining (6), (8), and (9) we obtain

$$1 \leqslant \frac{2nK_1}{qU} \|\,u\,\|_{1,A,\Omega} + \frac{1}{2} + \frac{K_1K_2}{U} \|\,u\,\|_{A,\Omega}\,,$$

from which it follows that

$$\|\,u\,\|_{A_*,\Omega} = U \leqslant K_3 \|\,u\,\|_{1,A,\Omega}\,. \tag{10}$$

Note that the constant K_3 depends only on n, A, and the cone C determining the cone property for Ω.

If $u \in W^1E_A(\Omega)$ then u is a norm limit (in that space) of a sequence of bounded functions with bounded supports, which sequence is, by (10), Cauchy in $E_{A_*}(\Omega)$. Thus (10) holds for u and $W^1E_A(\Omega) \rightarrow E_{A_*}(\Omega)$.

Now suppose $u \in W^1L_A(\Omega)$ is real-valued. Let

$$\begin{aligned} u_k(x) &= k && \text{if} \quad u(x) > k, \\ &= u(x) && \text{if} \quad -k \leqslant u(x) \leqslant k, \\ &= -k && \text{if} \quad u(x) < -k. \end{aligned}$$

By Lemma 4.2, $u_k \in W^1L_A(\Omega)$ and $\| u_k \|_{1,A,\Omega} \leqslant \| u \|_{1,A,\Omega}$. Let $\rho_k \in C^1([0, \infty))$ satisfy

(i) $0 \leqslant \rho_k(t) \leqslant 1$, $0 \leqslant \rho_k'(t) \leqslant 1$ for all $t \geqslant 0$,

(ii) $\rho_k(t) = 1$ if $t \leqslant k$, $\rho_k(t) = 0$ if $t \geqslant k + 2$.

Let $v_k(x) = \rho_k(| x |)\, u_k(x)$. Clearly v_k is bounded and has bounded support, and $v_k(x) \to u(x)$ a.e. in Ω as $k \to \infty$. Moreover, $\| v_k \|_{1,A,\Omega} \leqslant 2 \| u_k \|_{1,A,\Omega} \leqslant 2 \| u \|_{1,A,\Omega}$. By Fatou's lemma and (10),

$$\int_\Omega A_* \left(\frac{| u(x)|}{2K_3 \| u \|_{1,A,\Omega}} \right) dx \leqslant \liminf_{k\to\infty} \int_\Omega A_* \left(\frac{| v_k(x)|}{K_3 \| v_k \|_{1,A,\Omega}} \right),$$

whence $\| u \|_{A_*,\Omega} \leqslant 2K_3 \| u \|_{1,A,\Omega}$. This inequality extends to arbitrary elements of $W^1L_A(\Omega)$ by separate applications to real and imaginary parts. Thus $W^1L_A(\Omega) \to L_{A_*}(\Omega)$ and the proof is complete.

We must now prove Theorem 3.3(a) without the additional assumption that A satisfies (2). If A fails to satisfy (2) we need a new definition of A_*.

LEMMA 4.4. *Suppose that the N-function A satisfies*

$$\int_0^1 (A^{-1}(\tau)/\tau^{(n+1)/n})\, d\tau = \infty, \tag{11}$$

$$\int_1^\infty (A^{-1}(\tau)/\tau^{(n+1)/n})\, d\tau = \infty. \tag{12}$$

Then there exists a (nonunique) N-function $\hat{A}$ such that

(a) $\hat{A}(t) = A(t)$ *if* $t \geqslant A^{-1}(A(1)/2)$,

(b) $\int_0^1 (\hat{A}^{-1}(\tau)/\tau^{(n+1)/n})\, d\tau < \infty$,

(c) $\int_1^\infty (\hat{A}^{-1}(\tau)/\tau^{(n+1)/n})\, d\tau = \infty$.

(If, in addition, $A(t^{1/p})$ is an N-function for some p, $1 \leqslant p < n$, then $\hat{A}$ can be chosen to have the same property.)

Proof. Let r be chosen so that $1 < r < n$ (or $p < r < n$). There exists a point $t_0 < A^{-1}(A(1)/2)$ such that $A(t_0) < Kt_0^r$, where K is chosen so that $K[A^{-1}(A(1)/2)]^r = A(1)/2$; for, if not, we would have $A(t) \geqslant Kt^r$ for all t near zero, and hence $A^{-1}(t) \leqslant (t/K)^{1/r}$ for all small t, which contradicts (11). Let t_1 be the first point to the right of t_0 such that $A(t_1) = Kt_1^r$. Evidently $t_0 < t_1 \leqslant A^{-1}(A(1)/2)$. Let

$$\begin{aligned} \hat{A}(t) &= Kt^r && \text{if } t < t_1, \\ &= A(t) && \text{if } t \geqslant t_1. \end{aligned}$$

It is readily checked that $\hat{A}$ satisfies the requirements of the lemma. ∎

LEMMA 4.5. *Given any two N-functions A and B and a point $a > 0$, there exists a point b such that $0 < b < a$ and an N-function C such that $C(t) = A(t)$ if $t \geqslant a$ and $C(t) = B(t)$ if $t \leqslant b$.*

Proof. The right tangent to the graph $s = A(t)$ crosses the t-axis at a positive value of t. If this tangent intersects the graph of $s = B(t)$, let $t = b$ be the first such intersection point to the right of $t = 0$ and define

$$\begin{aligned} C(t) &= A(t) \qquad \text{if} \quad t \geqslant a, \\ &= B(t) \qquad \text{if} \quad t \leqslant b, \text{ rectilinear on } b \leqslant t \leqslant a. \end{aligned} \tag{13}$$

Otherwise there exists a unique line through the point $(a, A(a))$ which intersects and supports the convex set $\{(t, s): s \geqslant B(|t|)\}$. Let $t = b$ be any such intersection. Then $0 < b < a$ and C may again be defined by (13). ∎

DEFINITION 4.6. Suppose that A satisfies (11) and (12). In view of Lemma 4.4, the function $\hat{A}_*$ given by

$$\hat{A}_*(t) = \int_0^t (\hat{A}^{-1}(\tau)/\tau^{(n+1)/n})\, d\tau$$

is a well-defined N-function for which the conclusions of Lemma 4.1 are valid. By Lemma 4.5, an N-function A_* can now be constructed so that

$$\begin{aligned} A_*(t) &= \hat{A}_*(t) \qquad \text{if} \quad t \geqslant 1, \\ A_*(t) &= A(t) \qquad \text{if} \quad t \leqslant t_*, \end{aligned}$$

for some $t_* > 0$. Clearly there exists $K_* \geqslant 1$ such that

$$A_*(t) \leqslant K_* A(t), \tag{14}$$

for all $t \leqslant 1$.

LEMMA 4.7. *Lemma 4.3 remains valid even if (2) is not satisfied. In this case A_* must be defined as immediately above.*

Proof. Let $u \in W^1L_A(\Omega)$ be real-valued, and suppose that $\| u \|_{1,A,\Omega} = 1$. Let

$$\begin{aligned} u_1(x) &= 1 \qquad \text{if} \quad u(x) > 1, \\ &= u(x) \qquad \text{if} \quad |u(x)| \leqslant 1, \\ &= -1 \qquad \text{if} \quad u(x) < -1, \end{aligned}$$

and let $u_2 = u - u_1$. By Lemma 4.2, $u_1 \in W^1L_A(\Omega)$ and $\| u_1 \|_{1,A,\Omega} \leqslant 1$. Clearly also $\| u_2 \|_{1,A,\Omega} \leqslant 1$. Since (14) holds for $t \leqslant 1$, we have

$$\int_\Omega A_*\left(\frac{|u_1(x)|}{K_*}\right) dx \leqslant \int_\Omega A(|u_1(x)|)\, dx \leqslant 1,$$

whence $\| u_1 \|_{A_*,\Omega} \leqslant K_*$. We obtain a similar bound for $\| u_2 \|_{A_*,\Omega}$.

Let $\hat{\Omega} = \{x \in \Omega: |u(x)| > 1\} = \{x \in \Omega: |u_2(x)| > 0\}$. Then

$$1 \geqslant \int_\Omega A(|u(x)|)\, dx \geqslant \int_{\hat{\Omega}} A(|u(x)|)\, dx > A(1) \operatorname{vol} \hat{\Omega}.$$

If $\hat{\Omega}' = \{x \in \hat{\Omega}: |u_2(x)|/2 \leqslant A^{-1}(A(1)/2\}$ and $\hat{\Omega}'' = \hat{\Omega} \sim \hat{\Omega}'$, we have

$$\int_{\hat{\Omega}'} \hat{A}\left(\frac{|u_2(x)|}{2}\right) dx \leqslant \int_{\hat{\Omega}'} A\left(A^{-1}\left(\frac{A(1)}{2}\right)\right) dx \leqslant \frac{A(1)}{2} \operatorname{vol} \hat{\Omega} < \frac{1}{2},$$

and also

$$\int_{\hat{\Omega}''} \hat{A}\left(\frac{|u_2(x)|}{2}\right) dx \leqslant \frac{1}{2} \int_{\hat{\Omega}''} A(|u_2(x)|)\, dx \leqslant \frac{1}{2}.$$

Hence

$$\int_\Omega \hat{A}\left(\frac{|u_2(x)|}{2}\right) dx = \int_{\hat{\Omega}} \hat{A}\left(\frac{|u_2(x)|}{2}\right) dx \leqslant 1,$$

and so $\| u_2 \|_{\hat{A},\Omega} \leqslant 2$. Since derivatives of u_2 also vanish outside $\hat{\Omega}$, a similar estimate holds for them. Thus $\| u_2 \|_{1,\hat{A},\Omega} \leqslant 2$. By Lemma 4.3, $\| u_2 \|_{1,\hat{A},\Omega} \leqslant K_1$ for some constant K_1 independent of u. It follows that

$$\| u_2 \|_{A_*,\Omega} \leqslant K_2 = K_1 \left(1 + \frac{A_*(1)}{A(1)}\right).$$

In fact, denoting $\hat{\Omega}_1 = \{x \in \hat{\Omega}: |u_2(x)|/K_1 \geqslant 1\}$ and $\hat{\Omega}_2 = \hat{\Omega} \sim \hat{\Omega}_1$, and recalling that $A_*(t) = \hat{A}_*(t)$ for $t \geqslant 1$, we have

$$\begin{aligned}
\int_\Omega A_*\left(\frac{|u_2(x)|}{K_2}\right) dx &\leqslant \left(1 + \frac{A_*(1)}{A(1)}\right)^{-1} \left(\int_{\hat{\Omega}_1} + \int_{\hat{\Omega}_2}\right) A_*\left(\frac{|u_2(x)|}{K_1}\right) dx \\
&\leqslant \left(1 + \frac{A_*(1)}{A(1)}\right)^{-1} \left(\int_\Omega \hat{A}_*\left(\frac{|u_2(x)|}{K_1}\right) dx + A_*(1) \operatorname{vol} \hat{\Omega}\right) \\
&\leqslant 1.
\end{aligned}$$

We have now shown that $\| u \|_{A_*,\Omega} \leqslant K_* + K_2 = K_3$ independent of u satisfying $\| u \|_{1,A,\Omega} = 1$. For arbitrary real-valued $u \in W^1L_A(\Omega)$, it follows that

$$\| u \|_{A_*,\Omega} \leqslant K_3 \| u \|_{1,A,\Omega},$$

which in turn extends to complex-valued functions by separate applications to real and imaginary parts. ▌

Remark. Lemmas 4.3 and 4.7 together establish Theorem 3.3(a). Note that the function $A_*(t)$ as given by Definition 4.6 behaves in the same way for large values of t as does $A_*(t)$ given by (4). If $A_*(t)$ is given by (4), then (14) is satisfied as a consequence of Lemma 4.1, otherwise (14) has to be built into the definition of $A_*(t)$—for functions in $W^1L_A(\Omega)$ which are bounded but die off rather slowly at infinity cannot be expected to belong to $L_{A_*}(\Omega)$ if A_* does not satisfy (14). In spite of the nonunique way in which A_* is defined, the space $L_{A_*}(\Omega)$ is uniquely determined above.

Only part (b) of Theorem 3.3 remains to be proved. First we remark on the significance of the imbeddings

$$W^1E_A(\Omega) \to E_{A_*^{r/n}}(\Omega_r),$$
$$W^1L_A(\Omega) \to L_{A_*^{r/n}}(\Omega_r),$$

to be established. If $u \in W^1E_A(\Omega)$ then u is a norm limit of continuous functions ϕ of bounded support whose traces on Ω_r satisfy

$$\|\phi\|_{A_*^{r/n},\Omega_r} \leqslant K\|\phi\|_{1,A,\Omega}, \tag{15}$$

with K independent of ϕ. More generally, if $u \in W^1L_A(\Omega)$, then u is the limit pointwise a.e. of a sequence $\{v_k\}$ satisfying (15) and also $\|v_k\|_{1,A,\Omega} \leqslant 2\|u\|_{1,A,\Omega}$ (as in the proof of Lemma 4.3). Fatou's lemma then yields (15) for u with $2K$ in place of K.

We remark also that Lemma 4.1 assures us that $A_*^{r/n}$ is an N-function.

Proof of Theorem 3.3(b). The proof follows similar lines to that of Lemma 4.3. If $u \in W^1L_A(\Omega)$ is bounded and has bounded support, and does not vanish in $L_{A_*^{r/n}}(\Omega_r)$, then

$$\int_{\Omega_r}\left(A_*\left(\frac{|u(y)|}{U}\right)\right)^{r/n} dy = 1, \qquad \text{where} \quad U = \|u\|_{A_*^{r/n},\Omega_r}$$

Since we wish to show that

$$U \leqslant K\|u\|_{1,A,\Omega} \tag{16}$$

for some K independent of u, we may clearly assume that $\|u\|_{A,\Omega} \leqslant U$. Moreover, since (16) has already been proved for the special case $r = n$, we may also assume that $\|u\|_{A_*,\Omega} \leqslant U$.

Let $\omega(t) = [A_*(t)]^{1/q}$ where $q = np/(n-p)$ and let $f(x) = \omega(|u(x)|/U)$. It will become clear from estimates derived below that $f \in W^{1,p}(\Omega)$, which space is imbedded in $L^{rq/n}(\Omega_r)$ by the Sobolev imbedding theorem, Theorem 3.1. Hence

$$\begin{aligned}
1 &= \left\{\int_{\Omega_r}\left(A_*\left(\frac{|u(y)|}{U}\right)\right)^{r/n} dy\right\}^{(n-p)/r} = \|f\|^p_{rq/n,\Omega_r} \\
&\leqslant K_1\|f\|^p_{1,p,\Omega} \\
&= K_1\left\{\frac{1}{U^p}\sum_{j=1}^n\int_\Omega\left|\omega'\left(\frac{|u(x)|}{U}\right)\right|^p |D_ju(x)|^p\,dx + \int_\Omega\left|\omega\left(\frac{|u(x)|}{U}\right)\right|^p dx\right. \\
&\leqslant \frac{2K_1}{U}\sum_{j=1}^n\left\|\left(\omega'\left(\frac{|u|}{U}\right)\right)^p\right\|_{\tilde{B},\Omega}\|D_ju\|^p_{A,\Omega} + K_1\int_\Omega\left|\omega\left(\frac{|u(x)|}{U}\right)\right|^p dx.
\end{aligned} \tag{17}$$

In the last line above we have used the generalized Hölder inequality corresponding to complementary N-functions $\tilde{B}$ and B, and then the fact that $\| \, | v |^p \|_{B,\Omega} \leqslant \| v \|_{A,\Omega}^p$, since $B(t) = A(t^{1/p})$.

Now $B^{-1}(t) = [A^{-1}(t)]^p$ so, by (7),

$$[\omega'(t)]^p = \frac{1}{q^p} A_*(t) \frac{1}{B^{-1}(A_*(t))} \leqslant \frac{1}{q^p} \tilde{B}^{-1}(A_*(t)).$$

Since we are assuming $\| u \|_{A_*,\Omega} \leqslant U$ we have

$$\int_\Omega \tilde{B} \left(\left| \frac{\omega'(| u(x)|/U)}{1/q} \right|^p \right) dx \leqslant \int_\Omega A_* \left(\frac{| u(x)|}{U} \right) dx \leqslant 1$$

and hence

$$\|(\omega'(| u |/U))^p \|_{\tilde{B},\Omega} \leqslant q^{-p} \tag{18}$$

By Lemma 4.1 with $\epsilon = 1/2K_1$ we have, for some constant K_2 independent of u,

$$\begin{aligned} \int_\Omega \left| \omega \left(\frac{| u(x)|}{U} \right) \right|^p dx &\leqslant \frac{1}{2K_1} \int_\Omega A_* \left(\frac{| u(x)|}{U} \right) dx + K_2 \int_\Omega A \left(\frac{| u(x)|}{U} \right) dx \\ &\leqslant \frac{1}{2K_1} + K_2 \int_\Omega B \left[\left(\frac{\| u \|_{A,\Omega}}{U} \frac{| u(x)|}{\| u \|_{A,\Omega}} \right)^p \right] dx \\ &\leqslant \frac{1}{2K_1} + K_2 \frac{\| u \|_{A,\Omega}^p}{U^p} \int_\Omega A \left(\frac{| u(x)|}{\| u \|_{A,\Omega}} \right) dx \\ &\leqslant \frac{1}{2K_1} + \frac{K_2}{U^p} \| u \|_{A,\Omega}^p , \end{aligned} \tag{19}$$

since $\| u \|_{A_*,\Omega} \leqslant U$ and $\| u \|_{A,\Omega} \leqslant U$. Combining (17), (18), and (19) we have

$$1 \leqslant \frac{2nK_1}{q^p U^p} \| u \|_{1,A,\Omega}^p + \frac{1}{2} + \frac{K_1 K_2}{U^p} \| u \|_{A,\Omega}^p ,$$

from which it follows that

$$\| u \|_{A_*^{r/n},\Omega_r} = U \leqslant K_3 \| u \|_{1,A,\Omega} ,$$

with K_3 independent of u. The extension to more arbitrary functions u in $W^1E_A(\Omega)$ or $W^1L_A(\Omega)$ can now be carried out as in Lemma 4.3. ∎

5. Higher-Order Imbeddings

Given an N-function A we may construct a (terminating) sequence of N-functions $\{A_{*(j)} : j = 0, 1,..., M\}$ as follows

$$A_{*(0)} = A, \qquad A_{*(j)} = (A_{*(j-1)})_* .$$

(We recall that the Sobolev conjugate of any N-function satisfying $\int_1^\infty A^{-1}(\tau)\,\tau^{-(n+1)/n}\,d\tau = \infty$ is given either by (4) or by Definition 4.6 depending on whether $\int_0^1 A^{-1}(\tau)\,\tau^{-(n+1)/n}\,d\tau$ is finite or not). There will be a smallest integer M, depending on A, such that

$$\int_1^\infty \frac{A^{-1}_{*(M)}(\tau)}{\tau^{(n+1)/n}}\,dt < \infty. \tag{20}$$

In fact, $M \leqslant n$, for it is readily checked by induction on j that $A^{-1}_{*(j)}(t) \leqslant K_j t^{(n-j)/n}$ for large values of t.

THEOREM 5.1 (the Orlicz–Sobolev imbedding theorem). *Let Ω be a domain in R^n having the cone property. Let Ω_r denote the intersection of Ω with an r-dimensional plane in R^n. Let A be an N-function and let M be the smallest integer so that* (20) *holds.*

(a) *If $m \leqslant M$ then $W^mE_A(\Omega) \to E_{A_{*(m)}}(\Omega)$ and $W^mL_A(\Omega) \to L_{A_{*(m)}}(\Omega)$.*

(b) *If $m \leqslant M$, and if $\int_0^1 A^{-1}_{*(m-1)}(\tau)\,\tau^{-(n+1)/n}\,d\tau < \infty$, and if there exists p, $1 \leqslant p < n$, such that $B(t) = A_{*(m-1)}(t^{1/p})$ is an N-function, and if $n - p < r \leqslant n$ or $p = 1$ and $n - 1 \leqslant r \leqslant n$, then $W^mE_A(\Omega) \to E_{(A_{*(m)})^{r/n}}(\Omega_r)$ and $W^mL_A(\Omega) \to L_{(A_{*(m)})^{r/n}}(\Omega_r)$.*

(c) *If $m > M$, say $m = M + j$, then $W^mL_A(\Omega) \to C_B^{j-1}(\Omega)$. If Ω has the local Lipschitz property, then* (c) *can be strengthened to yield, for all $x, y \in \Omega$ and all α, $|\alpha| \leqslant j - 1$*

$$|D^\alpha u(x) - D^\alpha u(y)| \leqslant K\,\|u\|_{m,A,\Omega} \int_{|x-y|^{-n}}^\infty \frac{A^{-1}_{*(M)}(\tau)}{\tau^{(n+1)/n}}\,d\tau.$$

Each of these imbeddings is easily obtained by repeated applications of Theorem 3.3.

It should be remarked that all the imbeddings referred to in Theorem 5.1 (and also those in the theorems of Section 3) hold for arbitrary domains Ω provided the W space undergoing imbedding is replaced by the corresponding W_0 space ($W_0^{m,p}(\Omega)$ or $W_0{}^mL_A(\Omega)$).

6. Compact Imbeddings

THEOREM 6.1 (the Rellich–Kondrachov compactness theorem). *Let Ω be a bounded domain in R^n having the cone property and let Ω_r be the intersection of Ω with an r-dimensional plane in R^n. Then the following imbeddings are compact.*

$W^{m,p}(\Omega) \to L^q(\Omega)$ *if $mp < n$, $1 \leqslant q < np/(n - mp)$, or if $mp = n$, $1 \leqslant q < \infty$.*

$W^{m,p}(\Omega) \to L^q(\Omega_r)$ *if $0 < n - mp < r \leqslant n$, $1 \leqslant q < rp/(n - mp)$, or if $n = mp$, $1 \leqslant r \leqslant n$, $1 \leqslant q < \infty$.*

The appropriate generalization to imbeddings of Orlicz–Sobolev spaces was given by Donaldson and Trudinger in [5]. If A and B are N-functions, we say that B increases essentially more slowly that A near infinity if, for every number $k > 0$,

$$\lim_{t\to\infty} (B(kt)/A(t)) = 0.$$

THEOREM 6.2 (the Rellich–Kondrachov theorem for Orlicz spaces). *Let Ω, A, M be as specified in Theorem* 5.1 *and suppose in addition that Ω is bounded. Then the following imbeddings are compact.*

$W^m L_A(\Omega) \to E_B(\Omega)$	*if $m \leqslant M$ and B increases essentially more slowly than $A_{*(m)}$ near infinity.*
$W^m L_A(\Omega) \to E_B(\Omega_r)$	*if $m \leqslant M$, $A_{*(m-1)}(t^{1/p})$ is an N-function for some p such that $1 < p < n$ and $n - p < r \leqslant n$, and B increases essentially more slowly than $A_{*(m)}^{r/n}$ near infinity.*

Proofs of both of these theorems can be found in [2]. One does not in general expect compact imbeddings if the domain Ω is unbounded. However, certain imbeddings of the Sobolev spaces $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ are known to be compact for some unbounded domains Ω (see [1–3]), and these may be generalized in a straightforward manner to Orlicz–Sobolev spaces $W^m L_A(\Omega)$ and $W_0{}^m L_A(\Omega)$. An investigation of compact imbeddings of these latter spaces defined over unbounded domains has been carried out by Cahill in his thesis [4].

7. SOME UNANSWERED QUESTIONS

We conclude by calling the reader's attention to two areas in which the imbedding theory for Orlicz–Sobolev spaces is still incomplete.

The first gap concerns the "trace imbedding" Theorem 3.3(b) (and also 5.1(b)). As noted in Section 3, the assumption that $\int_0^1 A^{-1}(\tau)\, \tau^{-(n+1)/n}\, d\tau < \infty$ is not natural, and, even if Ω is unbounded, the theorem, like its special case, Theorem 3.3(a) (or 5.1(a)), ought to be provable without this assumption. The technique of Lemma 4.7 does not, however, lend itself to trace imbeddings and it is not clear whether or not 3.3(b) should be true if A_* is given as in Definition 4.6.

The second area of uncertainty concerns the sharpness of the Orlicz–Sobolev imbedding theorem. It is known that the target Lebesgue spaces for the imbeddings of $W^{m,p}(\Omega)$ given in Theorem 3.1(a) and (b) are "best possible" in the sense that the imbedding asserted cannot exist if q is larger than the upper endpoint of the specified interval. Moreover, Theorem 3.1(a) is best possible even if Orlicz spaces are allowed as possible target spaces. (Indeed, the author is grateful to the referee for pointing out that the noncompactness of the imbedding $W^{m,p}(\Omega) \to L^{np/(n-mp)}(\Omega)$ for bounded Ω with the cone property precludes the existence of an imbedding $W^{m,p}(\Omega) \to L_A(\Omega)$ if $A(t)$ increases essentially more rapidly near infinity than does $t^{np/(n-mp)}$.) However, are the imbeddings supplied by Theorems 3.2 (a) and (b) and 3.3 (a) and (b) best possible in the sense that the smallest possible Orlicz spaces have been used as target spaces? The answer cannot be "yes" in general, for if $A(t) = t^n/n$ then $A_*(t)$ is equivalent near infinity to $e^t - t - 1$, an N-function which increases essentially more slowly near infinity than the N-function $\exp(t^{n/(n-1)}) - 1$, which defines an Orlicz space into which $W^{1,n}(\Omega)$ can be imbedded if Ω is bounded and has the cone property. (This latter Orlicz space is known to be "best possible" target for $W^{1,n}(\Omega)$—see [7].) It is conjectured that Theorem 3.3(a) gives a "best possible" imbedding if A increases essentially more slowly near infinity than does the N-function t^p for some $p < n$.

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